THE DISCRETE SPECTRUM OF THE $D = 11$ BOSONIC $M5$-BRANE.

I. MARTIN [1], L. NAVARRO [2], A. J. PÉREZ A.[3] AND A. RESTUCCIA[4]

Abstract. We prove that the spectrum of the regularized $M5$-brane in $D = 11$ target space is discrete with eigenvalues extending to ∞. The proof includes the same result for the spectra of regularized bosonic p-branes in general.

1. Introduction

The understanding of the spectral properties of the supermembrane and super $M5$-brane in 11 dimensions are important steps towards the non-perturbative analysis of M-theory. The $SU(N)$ regularized Hamiltonian of the supermembrane on a $D = 11$ Minkowski target space has a continuous spectrum [1], see also [2], [3], [4], [5]. The supermembrane on a $D = 11$ target space with a compact sector is expected to have also a continuous spectrum [6]. But , the $D = 11$ supermembrane wrapped in an irreducible way on a compact sector of the target space, i.e., with a topological condition on configuration space yielding a non trivial central charge has a discrete spectrum and its ground state has a strictly positive energy [7], [8], [9], [10], [11], [12].

In all cases the bosonic Hamiltonian has a discrete spectrum. However, they are qualitatively different in a crucial way. In the latter case, the central charge generate mass terms implying that the potential on configuration space tends to infinity when it approaches infinity in this space. Moreover, this qualitative property of the spectrum remains unchanged in the supersymmetric theory with non trivial central charges. In the former case [1], the potential presents zero point valleys extending to infinity on configuration space. In this case the membrane admits as physical configurations stringy spikes that make the supermembrane spectrum continuous, despite the fact that its bosonic part on its own would not produce a continuous spectrum. Although the bosonic potential is zero on the minima of the valleys, the walls of the valleys get closer as they approach infinity in a way that the quantum mechanical wave function cannot escape to infinity. The precise mathematical meaning of this property was explained in [13]. It is formulated in terms of the integral of the potential on a fixed sized cell, defined in the sense of Molchanov and Maz'ya and Shubin [14], [15], when the center of the cell approaches infinity on configuration space. The potential integral of the cell in the directions of zero potential is bounded below by the potential of an harmonic oscillator ensuring that the integral goes to infinity when the cell is moved to infinity in those directions. This bound from below is lost once the fermionic part of the potential is added,

Date: compiled February 1, 2008.

as a consequence the supermembrane spectrum becomes continuous. All the above results refer to regularized Hamiltonians. More recently, it has been shown [10], [16] that the exact bosonic Hamiltonian for the case of the supermembrane with non trivial charges, has a discrete spectrum. This was achieved with a precise definition of the configuration space in terms of Sobolev spaces. It was also proven that the spectrum of the $SU(N)$ regularized model of the semiclassical Hamiltonian converges to the spectrum of the exact Hamiltonian when N tends to infinity.

In the case of the $M5$-brane no results have been reported concerning its spectrum. In [17] a semiclassical analysis of the spectrum of the $M5$-brane was performed. The $M5$-brane covariant action was first obtained in [18], [19] and a gauge fixed action version was obtained independently in [20]. In [21] a formulation was obtained for its Hamiltonian in terms of first class constraints only. Also, its BRST structure and the existence of string and membrane spikes were shown. Later, the Nambu-Poisson structure of the $M5$-brane was introduced in [22]. Also, a general analysis of such structure for p-branes was analyzed in [23]. In this paper, following [22], [13] we prove that the spectrum of the regularized $M5$-brane and that one of any p-brane is discrete. The proof in general applies to many matrix models associated to such theories. In section 2, the algebraic structure of the $M5$-brane Hamiltonian is presented for completeness. In section 3, we present the $SU(N)$ regularized version of the $M5$-brane Hamiltonian exploiting the Nambu-Poisson structure underlying it. In section 4, using appropriate theorems of spectral analysis, we show that generally p-branes matrix models present discrete spectra. With this result at hand, the discreteness of the spectrum of the regularized bosonic $M5$-brane Hamiltonian follows easily. In section 5, we present conclusions.

2. The Algebraic Structure of M5-Brane Hamiltonian

We start recalling the bosonic M5-Brane Hamiltonian on a $D = 11$ Minkowski target space in the light cone gauge obtained in [21],

$$\mathcal{H} = \frac{1}{2}\Pi^M\Pi_M + 2g + l^{\mu\nu}l_{\mu\nu} + \Theta_{5i}\Omega^{5i} + \Theta_j\Omega^j + \Lambda^{\alpha\beta}\Omega_{\alpha\beta} \tag{1}$$

where

$$l^{\mu\nu} = \frac{1}{2}(P^{\mu\nu} + {}^\star H^{\mu\nu}) \tag{2}$$

and

$$\begin{aligned} {}^\star H^{\mu\nu} &= \frac{1}{6}\epsilon^{\mu\nu\gamma\delta\lambda}H_{\gamma\delta\lambda} && (3)\\ H_{\gamma\delta\lambda} &= \partial_\rho B_{\lambda\sigma} + \partial_\sigma B_{\rho\lambda} + \partial_\lambda B_{\sigma\rho} && (4) \end{aligned}$$

Θ_{5i}, Θ_j, $\Lambda^{\alpha\beta}$ are the Lagrange multipliers associated to remaining constraints

$$\begin{aligned} \Omega^{5i} &= P^{5i} - {}^\star H^{5i} = 0 && (5)\\ \Omega^j &= \partial_\mu P^{\mu j} = 0, \quad i = 1,2,3,4 && (6) \end{aligned}$$

$$\Omega_{[\alpha\beta]} = \partial_{[\beta}\left[\frac{1}{\sqrt{W}}(\Pi_M\partial_{\alpha]}X^M + \frac{1}{4}V_{\alpha]})\right] = 0 \tag{7}$$

where

$$V_\mu = \epsilon_{\mu\alpha\beta\gamma\delta}l^{\alpha\beta}l^{\gamma\delta} \tag{8}$$

Π^M and $P^{\mu\nu}$ are the canonical conjugate momenta to X^M and $B_{\mu\nu}$ respectively, g is the determinant of the induced metric. Equations (5) and (6) are the first class constraints generating the local gauge symmetry associated to the antisymmetric field while (7) is the first class constraint generating volume preserving diffeomorphisms. X^M are the light cone gauge transverse coordinates on the target space.

W is a scalar density introduced in the gauge fixing procedure. It represents the determinant of an intrinsic metric over the spatial world volume of the brane. In our notation caps Latin letters are transverse light cone gauge indices $M, N = 1, \ldots, 9$, Greek ones are spatial world volume indices, and small Latin letters denote spatial world volume indices on a 4-dimensional spatial submanifold.

The elimination of second class constraints from the formulation in [18], [19] and [20] to produce a canonical Hamiltonian with only first class constraints, was achieved at the price of loosing the manifest 5 dimensional spatial covariance. In this way, the spatial world volume splits into $M_5 = M_4 \times M_1$. We will exploit that decomposition in our analysis of the Hamiltonian. We will assume M_4 has a symplectic structure with ω^0 being its associated non degenerate closed 2-form. It is assumed that M_4 and M_1 are compact manifolds.

Let us analyze the Hamiltonian density term by term. We first notice that g, the determinant of the induced metric, may be re-expressed in a straightforward manner as a squared five entries bracket, a Nambu-Poisson bracket,

$$\begin{aligned} g &= \frac{1}{5!}\epsilon^{\nu_1,\ldots,\nu_5}\epsilon^{\mu_1,\ldots,\mu_5} g_{\mu_1\nu_1}\cdots g_{\mu_5\nu_5} \\ &= \frac{1}{5!}\{X^M, X^N, X^P, X^Q, X^R\}^2. \end{aligned} \tag{9}$$

Let us consider now the third term dependent on the antisymmetric field $B_{\mu\nu}$. It is invariant under the action of the first class constraints (5) and (6). To eliminate part of these constraints, we proceed to make a partial gauge fixing on $B_{\mu\nu}$, following [21] we take

$$B_{5i} = 0 \tag{10}$$

which, together with the constraint (6) allow us a canonical reduction of the Hamiltonian (1). Notice that the contribution of this partial gauge fixing to the functional measure is 1. We are then left with the constraint

$$\partial_j P^{ij} + \partial_5 {}^\star H^{5i} = 0 \qquad i, j = 1, 2, 3, 4 \tag{11}$$

which generates the gauge symmetry on the 2-form B

$$\delta B_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i \tag{12}$$

B as a 2-form over M_4 may be decomposed using the Hodge decomposition theorem into an exact form plus a co-exact form plus an harmonic form. Its exact part is canonically conjugate to the co-exact part of P^{ij}, that is, calling $\partial_i \dot{b_j}$ the exact part of B_{ij} we have

$$\langle P^{ij}\partial_i \dot{b_j}\rangle = \langle -\partial_i P^{ij} \dot{b_j}\rangle \tag{13}$$

then an admissible gauge fixing is to set $b_j = 0$ to eliminate the exact form and the $\partial_i P^{ij}$ from the constraints.

We are then left with the co-exact part of B. It is directly related to l^{5i}

$$l^{5i} = \epsilon^{ijkl}\partial_j B_{kl}. \tag{14}$$

Noticing that l^{5i} is divergenceless, it may always be rewritten without loosing generality as

$$l^{5i} = \epsilon^{5ijkl}\partial_j \phi_{[a}\partial_k \phi_b \partial_l \phi_{c]}, \qquad a, b, c = 1, 2, 3. \tag{15}$$

This decomposition in terms of scalars is always valid locally for any four dimensional divergenceless smooth vectorial density. ϕ_a, $a = 1, 2, 3$ represent the three degrees of freedom of the co-exact part of B_{ij}.

Now we decompose the tensor density l^{ij} into

$$l^{ij} = \epsilon^{ji\alpha\beta\gamma}\partial_\alpha \phi_{[a}\partial_\beta \phi_b \partial_\gamma \phi_{c]} + \epsilon^{jikl}\omega_{kl} \tag{16}$$

where ω is a closed 2-form.

It is now possible, following the Darboux's theorem, to express ω_{kl} in terms of the canonical 2-form ω^0 over M_4. In fact the area preserving diffeomorphisms homotopic to the identity are generated by $\Omega_{\alpha\beta}$ with infinitesimal parameter $\xi^{\alpha\beta}$ or equivalently generated by

$$\Omega_\alpha = \left(\Pi_M \partial_\alpha X^M + \frac{1}{4}V_\alpha\right) \tag{17}$$

with infinitesimal parameter ξ^α given by

$$\xi^\alpha = \frac{1}{\sqrt{W}}\partial_\beta(\sqrt{W}\xi^{\alpha\beta}) \tag{18}$$

satisfying identically

$$\partial_\alpha(\sqrt{W}\xi^\alpha) = 0. \tag{19}$$

This volume preserving restriction leaves the four spatial parameters ξ^α associated to M_4 unconstrained. So, we are allowed to use the Darboux procedure to fix ω to ω^0. We should be left still with one free parameter since the local degrees of freedom of ω are only three. Indeed, that is the case since the corresponding gauge fixing procedure allows to eliminate l^{5i} from the constraints in the following way,

From (7) we have

$$\frac{1}{\sqrt{W}}\left(\Pi_M \partial_\alpha X^M + \frac{1}{4}V_\alpha\right) = \partial_\alpha U \tag{20}$$

where U is an auxiliary field. We notice in V_α, the product of the ϕ dependent terms is zero. In particular,

$$V_i = -4l^{5j}\omega^0_{ij} \tag{21}$$

it allows to eliminate l^{5j} from the equation for $\alpha = i$ in terms of U, which is determined from the equation for $\alpha = 5$:

$$l^{5j} = -\epsilon^{jikl}\omega^0_{kl}\partial_i U + \frac{\epsilon^{jikl}\omega^0_{kl}\Pi_M \partial_i X^M}{\sqrt{W}} \tag{22}$$

we are then left with the remaining constraint

$$\partial_j l^{5j} = \epsilon^{jikl}\omega^0_{kl}\partial_j\left(\frac{\Pi_M \partial_i X^M}{\sqrt{W}}\right) = 0. \tag{23}$$

The kinetic term associated to this gauge fixing is a total time derivative and, since ω^0 is time independent, it can be eliminated from the action. Finally, we are left with a complicated $l^{\mu\nu}l_{\mu\nu}$ term but to prove the discreteness of the spectrum it will become irrelevant as we will see in the following sections. The final gauge fixing

corresponding to the symplectomorphisms preserving ω^0 is performed by taking the Lagrange multiplier of the associated first class constraint to be zero, the ghosts fields decouple from the action.

3. Regularization of the M5-brane

After fixing ω to ω^0 we may resolve the volume-preserving constraint for ϕ^a $a = 1, 2, 3, 4$. We are then left still with one constraint,

$$\epsilon^{ijkl}\omega^0_{kl}\partial_i\left(\frac{\Pi_M\partial_j X^M}{\sqrt{W}}\right) = 0. \tag{24}$$

The left hand member generates the symplectomorphisms preserving ω^0. The full five dimensional diffeomorphisms have been reduced to only that generator. We are then left with a formulation in terms of X^M and its conjugate momenta Π_M, invariant under symplectomorphisms. The antisymmetric field $B_{\mu\nu}$ and its conjugate momenta $P^{\mu\nu}$ have been reduced to ω^0, there is no local dynamics related to them. All the dynamics may be expressed in terms of (X^M, Π_M).

In order to obtain a regularization of the Hamiltonian, we express $X^M(\tau,\sigma)$ and Π_M in terms of a complete orthonormal basis over $M_4 \times M_1$, $\{Y_a(\sigma^1, \sigma^2, \sigma^3, \sigma^4)\}$, in the Hilbert space of L^2 functions for M_4 and a Fourier basis for the M_1 manifold.

$$X^M(\tau,\sigma) = X^{a\,M}(\tau)Y_a(\sigma) \quad a = 1, 2, \ldots, \infty \tag{25}$$

$$\Pi_M(\tau,\sigma) = \sqrt{W}\Pi^a_M(\tau)Y_a(\sigma). \tag{26}$$

Since for every pair a, b

$$\frac{\epsilon^{ijkl}\omega^0_{kl}}{\sqrt{W}}\partial_i Y_a \partial_j Y_b \tag{27}$$

is a scalar function over M_4, we may reexpress it in terms of the basis, and obtain the symplectic bracket

$$\{Y_a, Y_b\} = \frac{\epsilon^{ijkl}\omega^0_{kl}}{\sqrt{W}}\partial_i Y_a \partial_j Y_b = f_{abc}Y_c \tag{28}$$

where f_{abc} is given by

$$\int_{M_4}\frac{\epsilon^{ijkl}\omega^0_{kl}}{\sqrt{W}}\partial_i Y_a \partial_j Y_b Y_c = f_{abc} \tag{29}$$

and satisfy the Jacobi identity. These are the structure constants of the symplectomorphisms preserving ω^0. Furthermore, we may introduce

$$Y_a Y_b = C_{abd}Y_d \tag{30}$$

it is again valid since $Y_a Y_b$ is also a scalar function over M_4. We get

$$\int_{M_4} Y_a Y_b Y_d = C_{abd} \tag{31}$$

which becomes totally symmetric in a, b, d. The other natural bracket in the formalism [22] is the Nambu one,

$$\{A, B, C, D, E\} = \frac{1}{\sqrt{W}}\epsilon^{\alpha\beta\gamma\delta\rho}\partial_\alpha A \partial_\beta B \partial_\gamma C \partial_\delta D \partial_\rho E \tag{32}$$

in particular the scalar

$$\{Y_a, Y_b, Y_c, Y_d, Y_e\} = f^g_{abcde}Y_g \tag{33}$$

where

$$\int_{M_5} \{Y_a, Y_b, Y_c, Y_d, Y_e\} Y_g = f_{abcdeg} \tag{34}$$

is a totally antisymmetric tensor satisfying a generalized Jacobi identity [24], [25], [26], [27]. By construction we have the following relation for the compact base manifold we are considering

$$f^{g}_{abcde} = \sum_{antisymm(a,b,c,d,e)} i n_a f^{\hat{c}}_{bc} f^{\hat{e}}_{de} C^{h}_{\hat{c}\hat{e}} C^{g}_{ha} \tag{35}$$

where $-n_a^2$ is an eigenvalue of the Laplacian on M_1. The right hand side satisfies the generalized Jacobi identity by construction.

We now consider a regularization of the $M5$-brane Hamiltonian by truncation of the infinite dimensional basis, that is, $a = 1, 2 \ldots . N$. We require in addition that in the remaining configuration space there exists brackets to have an intrinsic definition of the parameters f^c_{ab} and C_{abc} entering in the theory. In the large N limit the corresponding structure constants should be the area preserving ones. If so, we will have in the large N limit a generalized Jacobi identity for f^g_{abcde}. Meanwhile, it is not necessary to require a generalized Jacobi identity for the truncated theory. In the discreteness proof presented here we do not use any algebraic properties of the brackets. It is valid for any truncation in terms of some constants f^g_{abcde}. However, if we require an intrinsic meaning for the truncated theory, the algebraic structure should be present. We notice that the Nambu structure constants f^g_{abcde} of the symplectomorphisms satisfy the following properties

$$\int_{M_5} |\{Y_a, Y_b, Y_c, Y_d, Y_e\}|^2 = f_{abcdeg} f^{abcdeg} > 0$$

and more generally

$$M^{gh} = f^{g}_{abcde} f^{abcdeh} \tag{36}$$

is a strictly positive matrix. This is the only assumption we will require on the truncated theory for the associated f^g_{abcde}.

All the interacting terms in the Hamiltonian density may now be rewritten using $X^{a\,M}(\tau)$, their conjugate momenta and the structure constants f_{abc} and C_{abd}. Integrating on the spatial coordinates we arrive to a quantum mechanical Hamiltonian.

For the purpose of analyzing the spectrum of this regularized Hamiltonian, we will concentrate on the first interacting terms that do not include $l^{\mu\nu}$, this procedure is perfectly justified since

$$\mathcal{H} = \frac{1}{2}\Pi^M \Pi_M + 2g + l^{\mu\nu} l_{\mu\nu} \geq \frac{1}{2}\Pi^M \Pi_M + 2g = \mathcal{H}_0 \tag{37}$$

so, in what follows the spectrum of $\mathcal{H}_0$ is studied. We will prove that $\mathcal{H}_0$ has a discrete spectrum λ_n, with $\lambda_n \to \infty$ when $n \to \infty$. The min max theorem assures the same qualitative spectrum for $\mathcal{H}$ as it will be shown in the next section. The regularized Hamiltonian density $\mathcal{H}_0$ will correspond to a $p = 5$-brane.

4. The spectrum of the quantum mechanical Hamiltonian for regularized p-brane potentials

Let us consider the Schrödinger operator

$$H_L = -\Delta + V_L(X) = -\Delta + (X^{a_1}_{M_1} \dots X^{a_L}_{M_L} f^b_{a_1 \dots a_L})^2 \tag{38}$$

where L is the degree of the brane considered, $M_i = 1, \dots, K$, $a_i = 1, \dots, N$, $K \geq L$, $N \geq L$, $X = X^a_M \in \mathbb{R}^{KN}$ and $f^b_{a_1 \dots a_L}$ is a constant tensor totally antisymmetric in $a_1, \dots, a_L$ and it is not singular, i.e.,

$$\text{if} \quad X^{a_1}_M f^b_{a_1, a_2 \dots a_L} = 0 \quad \text{then} \quad X^{a_1}_M = 0. \tag{39}$$

We introduce

$$H_l = -\Delta + V_l(X) \qquad 1 \leq l \leq L \tag{40}$$

where

$$V_l = \sum_{\substack{M_1 \neq M_2 \neq \dots \neq M_l \\ 1 \leq M_i \leq K}} (X^{a_1}_{M_1} \dots X^{a_l}_{M_l} f^b_{a_1 \dots a_l b_{l+1} \dots b_L})(X^{\hat{a}_1}_{M_1} \dots X^{\hat{a}_l}_{M_l} f^b_{\hat{a}_1 \dots \hat{a}_l b_{l+1} \dots b_L}).$$

In this section we prove that H_l, $1 \leq l \leq L$ has a discrete spectrum, but first let us recall some mathematical definitions and theorems needed.

The definition of capacity of a compact set in $\mathbb{R}^n$ is an important ingredient in the Maz'ya and Shubin generalization of Molchanov's theorem [14] on the necessary and sufficient conditions for the discreteness of the spectrum of the Schrödinger operator. For details and the full-general version of the Maz'ya and Shubin theorem see [15].

Definition 1. Let $n \geq 3$, $F \subset \mathbb{R}^n$ be a compact, and $Lip_c(\mathbb{R}^n)$ the set of all real-valued functions with compact support satisfying a uniform Lipschitz condition in $\mathbb{R}^n$. The Wiener's capacity of F is defined by

$$\operatorname{cap}(F) = \operatorname{cap}_{\mathbb{R}^n}(F) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \,\middle|\, u \in Lip_c(\mathbb{R}^n),\ u|_F = 1 \right\}.$$

In physical terms the capacity of the set $F \subset \mathbb{R}^n$ is defined as the electrostatic energy over $\mathbb{R}^n$ when the electrostatic potential is set to 1 on F.

Definition 2. Let $\mathcal{G}_d \subset \mathbb{R}^n$ be an open and bounded star-shaped set of diameter d, let $\gamma \in (0,1)$. The *negligibility class* $\mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$ consists of all compact sets $F \subset \overline{\mathcal{G}_d}$ satisfying $\operatorname{cap}(F) \leq \gamma \operatorname{cap}(\overline{\mathcal{G}_d})$.

For example we can take $\mathcal{G}_d$ to be a n-cube or a ball in $\mathbb{R}^n$. In what follow we refer to $\mathcal{G}_d \setminus F$ as a cell.

Theorem 1 (Maz'ya and Shubin)**.** *Let $V \in L^1_{loc}(\mathbb{R}^n)$, $V \geq 0$.*

Necessity: If the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^n)$ is discrete then for every function $\gamma : (0, +\infty) \to (0,1)$ and every $d > 0$

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)} \int_{\mathcal{G}_d \setminus F} V(x) \, dx \to +\infty \quad \text{as} \quad \mathcal{G}_d \to \infty. \tag{41}$$

Sufficiency: Let a function $d \mapsto \gamma(d) \in (0,1)$ be defined for $d > 0$ in a neighborhood of 0 and satisfying

$$\limsup_{d \downarrow 0} d^{-2}\gamma(d) = +\infty.$$

Assume that there exists $d_0 > 0$ such that (41) holds for every $d \in (0, d_0)$. Then the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^n)$ is discrete.

Remark 2. *It follows from the previous theorem that a necessary condition for the discreteness of spectrum of $-\Delta + V$ is*

$$\int_{\mathcal{G}_d} V(x)\,dx \to \infty \quad \textit{as} \quad \mathcal{G}_d \to \infty. \tag{42}$$

Let us recall that K. Friedrichs (see [15] for further references) proved that the spectrum of the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^n)$ with a locally integrable potential V is discrete provided $V(x) \to \infty$ as $|x| \to \infty$.

In what follow we will denote the n-dimensional Lebesgue measure by $\mathrm{Vol}(\cdot)$.

Lemma 3. *For each given ball $\mathcal{G}_d = \mathcal{G}_d(x_0)$ centered at x_0 and radius $d > 0$.*

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d;\mathbb{R}^n)} \mathrm{Vol}(\mathcal{G}_d \setminus F) > 0.$$

Proof. Let $V(x) = |x|$. Then by Friedrichs theorem the spectrum of $-\Delta + V$ is discrete, so by Theorem 1 we have

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d;\mathbb{R}^n)} \int_{\mathcal{G}_d \setminus F} V(x)\,dx \to \infty \quad \text{as} \quad |x_0| \to \infty.$$

Now $\int_{\mathcal{G}_d \setminus F} V(x)\,dx \le (|x_0| + d)\,\mathrm{Vol}(\mathcal{G}_d \setminus F)$ implies that

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d;\mathbb{R}^n)} \int_{\mathcal{G}_d \setminus F} V(x)\,dx \le (|x_0| + d) \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d;\mathbb{R}^n)} \mathrm{Vol}(\mathcal{G}_d \setminus F)$$

from which follows that $\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d;\mathbb{R}^n)} \mathrm{Vol}(\mathcal{G}_d \setminus F) > 0$, as we claimed. □

The min-max principle is useful in the proof of the next proposition, so we state it for completeness.

Theorem 4 (Min-max principle)**.** *Let H be a self-adjoint operator that is bounded from below, i.e, $H \ge cI$ for some c. Define*

$$\mu_n(H) = \sup_{\phi_1, \phi_2 \ldots \phi_{n-1}} U_H(\phi_1, \phi_2 \ldots \phi_{n-1})$$

where

$$U_H(\phi_1, \phi_2 \ldots \phi_m) = \inf(\psi, H\psi) \quad \textit{when} \quad \|\psi\| = 1 \quad \textit{and} \quad \psi \in [\phi_1, \phi_2 \ldots \phi_m]^\perp$$

$[\phi_1, \phi_2 \ldots \phi_m]^\perp$ is shorthand for $\{\psi | (\psi, \phi_i) = 0, i = 1, 2, \ldots, m\}$. The ϕ_i are not necessarily independent.

Then, for each fixed n, either:

(a) there are n eigenvalues (counting degenerate eigenvalues a number of times equal to there multiplicity) below the bottom of the essential spectrum, and $\mu_n(H)$ is the nth eigenvalue counting multiplicity;

or

(b) μ_n is the bottom of the essential spectrum, i.e., $\mu_n = \inf\{\lambda | \lambda \in \sigma_{ess}(H)\}$, and in that case $\mu_n = \mu_{n+1} = \mu_{n+2} = \ldots$ and there are at most $n-1$ eigenvalues (counting multiplicity) below μ_n.

For proof of the min-max theorem and further reading on the subject see [28].

Remark 5. *As a consequence of this theorem if A and B are self-adjoint operators bounded from below and if $A \leq B$, then $\mu_n(A) \leq \mu_n(B)$. In this case if $\mu_n(A) \to \infty$ when $n \to \infty$ then $\mu_n(B)$ will also tend to infinity. This, in turn, means that if A has a discrete spectrum with a compact resolvent then B will also have a discrete spectrum with a compact resolvent.*

Now, we prove a theorem concerning the above defined operator H_l. In what follow we use DS for discrete spectrum.

Proposition 6. *Let*

$$V_l = \sum_{1 \leq M_i \leq K} (X^{a_1}_{M_1} \dots X^{a_l}_{M_l} f^b_{a_1 \dots a_l b_{l+1} \dots b_L})^2$$

with $M_1 \neq M_2 \neq \dots \neq M_l$.

Then the following sequence holds:

$$H_l \text{ has DS} \Rightarrow -\Delta + \sqrt{V_l} \text{ has DS} \Rightarrow H_{l+1} \text{ has DS.}$$

Proof. **A.** *H_l has DS $\Rightarrow -\Delta + \sqrt{V_l}$ has DS.*

Let $\mathcal{G}_d = \mathcal{G}_d(X_0) \subset \mathbb{R}^{KN}$ be a ball centered at X_0 and radius $d > 0$, let $F \in \mathcal{N}_\gamma(\mathcal{G}_d; \mathbb{R}^n)$. Then $X = X_0 + \xi$ for all X in the cell $\mathcal{G}_d \setminus F$. Let Ω_F be the set of all such ξ. Then the necessary condition of Theorem 1 implies that

$$\inf_F \int_{\Omega_F} V_l(X_0, \xi)\, d\xi \to \infty \quad \text{as} \quad |X_0| \to \infty \tag{43}$$

We can rewrite the potential as

$$V_l = \sum_{j=1}^{N_l} P_j^2(X_0, \xi) \tag{44}$$

where $P_j,\ j = 1, \dots, N_l$, are polynomials in ξ with coefficients depending on X_0. Using the Gram-Schmidt process it is possible to rewrite

$$P_j(X_0, \xi) = \sum_{k=1}^{n_j} a_{jk}(X_0) \varphi_{jk}(\xi)$$

where $\{\varphi_{jk}(\xi)\}$ is a finite system of orthonormal polynomials depending on Ω_F i.e.,

$$\int_{\Omega_F} \varphi_{jk}(\xi) \varphi_{im}(\xi)\, d\xi = \delta_{(j,k)(i,m)} \quad \text{(the Kroneker delta).}$$

and $a_{jk}(X_0)$ are its corresponding coefficients.

Note that for any system $\{\varphi_{jk}\}$ there exists $M_F > 0$ such that $|\varphi_{jk}(\xi)| \leq M_F$ for all (j,k) and all $\xi \in \Omega_F$.

Hence

$$\int_{\Omega_F} P_j^2(X_0, \xi) = \sum_{k=1}^{n_j} a^2_{jk}(X_0)_{\Omega_F} =: \|P_j\|^2_{\Omega_F} \text{ and } \int_{\Omega_F} V_l\, d\xi = \sum_j \|P_j\|^2_{\Omega_F} \tag{45}$$

It is possible to choose $N_0 \geq \max\{N_l, n_j : j = 1, \ldots, N_l\}$ independent of F, then using $\left(\sum_{k=1}^{n} a_k\right)^2 \leq n \sum_{k=1}^{n} a_k^2$ twice, we have $P_j^4 \leq N_0^3 \sum_{k=1}^{n_j} a_{jk}^4 \varphi_{jk}^4(\xi)$, therefore

$$\int_{\Omega_F} P_j^4\, d\xi \leq N_0^3 \sum_{k=1}^{n_j} a_{jk}^4 \int_{\Omega_F} \varphi_{jk}^4(\xi)\, d\xi \leq N_0^3 M_F^2 \sum_{k=1}^{n_j} a_{jk}^4 \leq N_0^3 M_F^2 \left(\sum_{k=1}^{n_j} a_{jk}^2\right)^2$$

i.e. $\int_{\Omega_F} P_j^4\, d\xi \leq N_0^3 M_F^2 \|P_j\|_{\Omega_F}^4$. Then from this and (44) and (45)

$$\int_{\Omega_F} V_l^2(X_0,\xi)\, d\xi \leq N_0^4 M_F^2 \sum_j \|P_j\|_{\Omega_F}^4 \leq N_0^4 M_F^2 \left(\int_{\Omega_F} V_l(X_0,\xi)\, d\xi\right)^2 \tag{46}$$

Because of $V_l^\alpha \in L^2(\Omega_F)$ for all $\alpha \geq 0$, using Schwarz inequality twice we obtain:

$$\left(\int_{\Omega_F} V_l\, d\xi\right)^{3/2} \leq \int_{\Omega_F} V_l^{1/2}\, d\xi \left(\int_{\Omega_F} V_l^2\, d\xi\right)^{1/2} \tag{47}$$

Now using (46) and (47) we have:

$$\left(\int_{\Omega_F} V_l\, d\xi\right)^{1/2} \leq N_0^2 M_F \int_{\Omega_F} V_l^{1/2}\, d\xi$$

And from (43) we concluded that

$$\inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d;\mathbb{R}^n)} M_F > 0 \text{ and } -\Delta + \sqrt{V_l} \text{ has DS.}$$

B. $-\Delta + \sqrt{V_l}$ *has DS* $\Rightarrow H_{l+1}$ *has DS* .

Now we show that $-\Delta + V_{l+1}$ also has a DS using the min-max principle. We start with $-\Delta + V_{l+1}$ and get a bound in terms of $-\Delta + \sqrt{V_l}$. We rewrite V_{l+1} as

$$V_{l+1} = \sum_{M_1 \neq M_2 \neq \ldots \neq M_{l+1}} (X_{M_1}^{a_1} \ldots X_{M_{l+1}}^{a_{l+1}} f^b_{a_1 \ldots a_{l+1} b_{l+2} \ldots b_L})(X_{M_1}^{\hat{a}_1} \ldots X_{M_{l+1}}^{\hat{a}_{l+1}} f^b_{\hat{a}_1 \ldots \hat{a}_{l+1} b_{l+2} \ldots b_L})$$
$$= \sum_M \left[\sum_{M_1 \neq M_2 \neq \ldots \neq M} (X_{M_1}^{a_1} \ldots X_{M_l}^{a_l} f^b_{a_1 \ldots a_l c b_{l+2} \ldots b_L})(X_{M_1}^{\hat{a}_1} \ldots X_{M_l}^{\hat{a}_l} f^b_{\hat{a}_1 \ldots \hat{a}_l \hat{c} b_{l+2} \ldots b_L}) \right] X_M^c X_M^{\hat{c}}$$

and notice that it is a sum of harmonic oscillator potentials with a matrix coefficient not involving X_M. Let $\Delta_M = \sum_{c=0}^{N} \frac{\partial^2}{(\partial X_M^c)^2}$, then for any k, $0 < k < 1$, we have

$$-\Delta + V_{l+1} = (1-k)(-\Delta) + \sum_M [-k\Delta_M +$$
$$\sum_{M_1 \neq \ldots \neq M_l \neq M} (X_{M_1}^{a_1} \ldots X_{M_l}^{a_l} f^b_{a_1 \ldots a_l c \ldots b_L})(X_{M_1}^{\hat{a_1}} \ldots X_{M_l}^{\hat{a_l}} f^b_{\hat{a_1} \ldots \hat{a_l} \hat{c} \ldots b_L}) X_M^c X_M^{\hat{c}}]$$
$$= (1-k)(-\Delta) + k \sum_M \left(-\Delta_M + \frac{1}{k} A_{c\hat{c}}^{(M)} X_M^c X_M^{\hat{c}}\right) \tag{48}$$

$$\geq (1-k)(-\Delta) + \sum_M \sqrt{k}\sqrt{\operatorname{tr} A^{(M)}} \geq (1-k)\left[-\Delta + \frac{\sqrt{k}}{(1-k)}\sqrt{V_l}\right]$$

By hypothesis $\sqrt{V_l}$ satisfies the sufficient condition of Theorem 1, then so does $\frac{\sqrt{k}}{(1-k)}\sqrt{V_l}$, hence the righthand side of (48) has a DS. It implies that $-\Delta + V_{l+1}$ also has a DS by Theorem 4. So the proposition is proved. □

Remark 7. If $-\Delta + \sqrt{V_l}$ has DS then $-\Delta + V_l$ has DS if V_l is locally bounded i.e., for all compact set K there exists $M_K > 0$ such that $V_l(X) \leq M_K$ for all $X \in K$, because of Lemma 3 and

$$\left(\int_{\mathcal{G}_d \setminus F} \sqrt{V_l}\, dX\right)^2 \leq \operatorname{Vol}(\mathcal{G}_d \setminus F) \int_{\mathcal{G}_d \setminus F} \sqrt{V_l}\, dX$$

With the latter proposition proved it is straightforward to show that

Proposition 8. *H_l has a discrete spectrum, for all $1 \leq l \leq L$.*

Proof. H_1 is the Hamiltonian for a harmonic oscillator, it then has a DS. One makes use of the sequence proved in Proposition 6 to conclude that H_l has a DS. □

To show that the regularized $M5$-brane has a discrete spectrum with eigenvalues tending to infinity, we need to prove first that the p-brane has the maximum of its eigenvalues going to infinity, since the proof of discreteness is not enough to prove a compact resolvent.

Proposition 9. *The spectrum of H_l with $1 \leq l \leq L$ has eigenvalues λ_n satisfying*

$$\lambda_n \to \infty \qquad \textit{when} \qquad n \to \infty. \tag{49}$$

Proof. H_1 is the Hamiltonian for a harmonic oscillator , it has a compact resolvent, hence its spectrum satisfies (49). We now consider any of the potentials $\sqrt{V_l}$ in the sequence of Proposition 6. Let us denote it V. It is of the form

$$V(X) = \sqrt{R^n}W(\hat{\theta}) \qquad \text{where} \qquad \|W(\hat{\theta})\| = 1 \tag{50}$$

R, $\hat{\theta}$ are polar coordinates. We consider the neighborhood Ω_ϵ of zeros of $W(\hat{\theta})$, i.e.,

$$\Omega_\epsilon = \left\{\hat{\theta} : W(\hat{\theta}) < \epsilon\right\} \qquad\qquad \epsilon > 0. \tag{51}$$

We then define $V_\epsilon(X) = \sqrt{R^n}W_\epsilon(\hat{\theta})$ where

$$W_\epsilon(\hat{\theta}) = W(\hat{\theta}) \qquad\qquad X \in \text{complement of } \Omega_\epsilon \tag{52}$$

$$W_\epsilon(\hat{\theta}) = \epsilon \qquad\qquad X \in \Omega_\epsilon. \tag{53}$$

For any $\epsilon > 0$, $V_\epsilon(X) \to \infty$ as $|X| \to \infty$, hence the spectrum of $H_\epsilon = -\Delta + V_\epsilon$ satisfy (49).

In fact, using the min-max theorem is easy to see that if we define a Hamiltonian $H_{well} = -\Delta + W$, where W is a potential well $-c$ constant in the region inside a ball S and 0 outside it. And the ball S is taken in such a way that $V_\epsilon \geq c$ outside it (this is always possible since $V_\epsilon(X) \to \infty$ as $|X| \to \infty$) then $V_\epsilon \geq c + W$ so that $\mu_n(H_\epsilon) \geq c + \mu_n(H_{well})$. Using the fact that $\mu_n(H_{well}) \geq -1$ with $n \geq N$, for some N, since W is a bounded potential of compact support, we get $\mu_n(H_\epsilon) \geq c - 1$ if $n \geq N$ but since c is arbitrary $\mu_n(H_\epsilon) \to \infty$ as $n \to \infty$. So, the spectrum of H_ϵ satisfy (49).

When $\epsilon \to 0$, the spectrum of H_ϵ could become continuous but we already know from Proposition 6 that it is discrete. It then follows property (49) also for the case when $\epsilon \to 0$. □

Proposition 10. *The Hamiltonian of the regularized bosonic $M5$-brane H satisfies property* (49).

Proof. We have $H > H_4$, from the above propositions H_4 has a discrete spectrum satisfying property (49). Theorem 4 ensures then $\mu_n(H) > \mu_n(H_4)$ so the spectrum of H also satisfies (49). H has a compact resolvent in the Hilbert space obtained by the completion of the subspace generated by its eigenfunctions. □

5. Conclusions

We showed the discreteness of the spectrum of the $M5$-brane and, in general, of p-brane theories. The condition is obtained from the Molchanov, Maz'ya and Shubin necessary and sufficient condition on the potential of a Schrödinger operator to have a discrete spectrum. The criteria is expressed not in terms of the behaviour of the potentials at each point, but by a mean value, on the configuration space. The mean value in the sense of Molchanov considers the integral of the potential on a finite region of configuration space. It can be naturally associated to a discretization of configuration space in the quantum theory. We found that the mean value in the direction of the valleys where the potential is zero, at large distances in the configuration space, is the same as that of a harmonic oscillator. Also, using the min-max principle it was shown that the discrete spectra had eigenvalues running to infinity showing that their respective resolvent operators are compact.

Acknowledgments

A. R. would like to thank Perimeter Institute for kind hospitality . Also, A. R. and I. M. are grateful to E. Planchart and V. Strauss (USB), L. Boulton (Heriot-Watt U.), F. Cachazo (Perimeter Institute) and M. P. Garcia del Moral (Turin U.) for fruitful discussions. This work was supported by PROSUL under contract CNPq 490134/2006-8 and Decanato de Investigaciones y Desarrollo(DID-USB), Proyecto G-11.

References

[1] B. de Wit, M.Luscher, H. Nicolai. Nucl.Phys.**B320**,(1989),135.
[2] H Nicolai, R Helling. `hep-th/9809103`
[3] M. Lüscher. Nucl.Phys.**B219** (1983) 233-261.
[4] B. de Wit, J. Hoppe, H. Nicolai. Nucl.Phys. **B305**:(1988) 545.
[5] B. de Wit, Marquard and H. Nicolai. Commun.Math.Phys. **128** (1990)39-62.
[6] B. de Wit, K. Peeters, J. Plefka. Phys.Lett.**B409**, (1997),117-123.
[7] M.P.Garcia del Moral, A. Restuccia. Phys.Rev. **D66** (2002) 045023 `hep-th/0103261`
[8] L. S. Boulton, A. Restuccia. Nucl.Phys. **B724** (2005)380-396.
[9] L. S. Boulton, M. P. Garcia del Moral, I. Martin, A. Restuccia. Class.Quant.Grav. **19** (2002) 2951. `hep-th/0109153`
[10] L. Boulton, M.P.Garcia del Moral, A. Restuccia. Nucl.Phys. **B671** (2003) 343-358, `hep-th/0211047`
[11] J. Bellorin, A. Restuccia . Nucl.Phys. **B737** (2006) 190-208,`hep-th/0510259`
[12] I.Martin and A.Restuccia, Nucl. Phys. **B622**, (2002) 240 ,hep-th/0108046.
[13] M.P.Garcia del Moral, L. Navarro, A.J. Perez, A. Restuccia. Nucl.Phys. **B 765** (2007) 287-298 `hep-th/0607234`
[14] A.M.Molchanov. Proc. Moscow Math. Society **2** (1953) 169-199 (Russian).

[15] Mazy'a, V. and Shubin, M. Annals of Mathematics, **162** (2005), 919-942.
[16] L. Boulton, M.P.Garcia del Moral, A. Restuccia. hep-th/0609054.
[17] J. Maldacena, M. M. Sheikh-Jabbari, M. Van Raamsdonk. hep-th/0211139.
[18] D.S.Paolo Pasti and M.Tonin,Phys. Lett. **B398**,(1997)41 .
[19] I.Bandos, K.Lechner,A.Nurmagambetov,P.Pasti,D.Sorokin,and M.Tonin, Phys.Rev. Lett. **78**,(1997)4332 .
[20] C.P. M.Aganagic, J.Park and J.Schwarz, Nucl. Phys. B **496**,(1997) 191 .
[21] A.De Castro and A.Restuccia,Nucl. Phys. **B617**,(2001) 215,hep-th/0103123.
[22] A.De Castro, M.P.Garcia del Moral, I. Martin, A.Restuccia,Phys. Lett. **B 584**,(2004),171-177.
[23] J.Hoppe,Helv.Phys.Acta **70**,(1997) 302.
[24] Y.Nambu, Phys. Rev. **D7**, (1973)2405.
[25] L.Takhtajan, Commun. Math. Phys. **160**,295 (1994),hep-th/9301111.
[26] J.Grabowski and G.Marmo(1999a), math.DG/9902127.
[27] J.Grabowski and G.Marmo (1999b),math.DG/9902128.
[28] M.Reed and B.Simon, *IV: Analysis of Operators* Methods of Modern Mathematical Physics, Academic Press,1978.

[1] Departamento de Física, Universidad Simón Bolívar, Apartado 89000, Caracas 1080-A, Venezuela.
E-mail address: isbeliam@usb.ve

[2] Departamento de Matemáticas, Universidad Simón Bolívar, Apartado 89000, Caracas 1080-A, Venezuela.
E-mail address: lnavarro@ma.usb.ve

[3] Departamento de Matemáticas, Universidad Simón Bolívar, Apartado 89000, Caracas 1080-A, Venezuela.
E-mail address: ajperez@ma.usb.ve

[4] Departamento de Física, Universidad Simón Bolívar, Apartado 89000, Caracas 1080-A, Venezuela.
E-mail address: arestu@usb.ve